

Constructive Stabilization and Pole Placement by Arbitrary Decentralized Architectures

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Abstract—A seminal result in decentralized control is the development of fixed modes by Wang and Davison in 1973 - that plant modes which cannot be moved with a static decentralized controller cannot be moved by a dynamic one either, and that the other modes which can be moved can be shifted to any chosen location with arbitrary precision. These results were developed for perfectly decentralized, or block diagonal, information structure, where each control input may only depend on a single corresponding measurement. Furthermore, the results were claimed after a preliminary step was demonstrated, omitting a rigorous induction for each of these results, and the remaining task is nontrivial.

In this paper, we consider fixed modes for arbitrary information structures, where certain control inputs may depend on some measurements but not others. We provide a comprehensive proof that the modes which cannot be altered by a static controller with the given structure cannot be moved by a dynamic one either, and that the modes which can be altered by a static controller with the given structure can be moved by a dynamic one to any chosen location with arbitrary precision, thus generalizing and solidifying Wang and Davison's results.

This shows that a system can be stabilized by a linear time-invariant controller with the given information structure as long as all of the modes which are fixed with respect to that structure are in the left half-plane; an algorithm for synthesizing such a stabilizing decentralized controller is then distilled from the proof.

Index Terms—Network Analysis and Control, Decentralized Control, Stability of Linear Systems, Linear Systems

I. INTRODUCTION

This paper is concerned with the stabilization of decentralized control systems, for which certain controller inputs may depend on some measurements but not others. This corresponds to finding a stabilizing controller which satisfies a given sparsity constraint. A special case of this, sometimes referred to as *perfectly decentralized control*, occurs when each control input may depend only on a single associated measurement, which corresponds to finding a stabilizing controller which is (block) diagonal.

This special case is sometimes itself referred to as *decentralized control*, particularly in the literature from a few decades ago. This malleability or evolution of the definition has not only caused some confusion, but has also resulted in some

important results in the field only being studied for this special case.

We assume that plants and controllers are finite-dimensional, linear time-invariant (FDLTI), except for when we say otherwise.

A seminal result in decentralized control is the development of *fixed modes* by Wang and Davison in 1973 [1]. This paper studied FDLTI perfectly decentralized stabilization of FDLTI systems. Its contributions can be broken into three main components - a definition establishing the framework, and two subsequent results. Fixed modes were defined as those modes of the plant which could not be altered by any static perfectly decentralized controller (that is, by any diagonal matrix). The first result was that these fixed modes could also not be altered by any dynamic perfectly decentralized controller; if you can't move it with a static diagonal controller, you can't move it with a dynamic diagonal controller. The second result was that if a mode is not fixed, then it can be moved arbitrarily close to any chosen location in the complex plane (provided that it has a complex conjugate pair if it is not real). These can be taken together to state that a system is stabilizable by a (dynamic) perfectly decentralized controller if and only if all of its (static) fixed modes are in the left half-plane (LHP).

When proving these results, it was shown that allowing one part of the controller to be dynamic does not result in any fewer fixed modes than a static controller, and then claimed that the first result followed; that is, that a dynamic controller would not be able to move any of the fixed modes. Similarly, it was shown that a single non-fixed mode could be moved to any chosen location, and then claimed that the second result followed; that is, that an arbitrary number of non-fixed modes could be simultaneously moved to chosen locations by a single controller. Getting from these initial steps to a rigorous inductive argument, however, is not trivial.

We seek to study these fundamental concepts for arbitrary information structure, while developing robust notation and rigorous proofs, thus placing the new and existing results on a sound mathematical footing.

We first introduce notation for fixed modes that allows it to vary with information structure, as well as with the type of controllers allowed (static, dynamic, linear, etc.). We then show that, for arbitrary information structure, the fixed modes with respect to dynamic controllers are the same as the fixed modes with respect to static controllers. Moreover, we provide a rigorous proof that the non-fixed modes can then be moved to within an arbitrarily small distance of chosen (conjugate) locations, using a dynamic LTI controller with the given structure, thus extending and solidifying the seminal results of

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Wang and Davison. The proof is constructive, and we lastly distill an explicit algorithm for the stabilizing decentralized controller synthesis from the proof.

The obvious potential benefits of this are an increased understanding of decentralized stabilizability, and the verification of important existing results. It is also our hope that the notation developed will be useful in further extending our understanding of decentralized stabilizability to richer classes of controllers for which the fixed modes may diminish relative to the original static definition, particularly non-linear and/or time-varying controllers [2]–[5]. We further note that demonstrating the results of this paper directly for arbitrary structure, as opposed to attempting to diagonalize the problem and then prove the original perfectly decentralized results, would likely be useful when other types of stability are required which are not invariant under such transformations, such as bounded-input bounded-output (BIBO) stability, though we currently focus on internal state stability. As an example of the diagonalization approach, readers are referred to [6], where existence of a stabilizing controller under arbitrary information constraint has been demonstrated by transforming the problem into a diagonal one to which [1] could be applied. Furthermore, [6] demonstrates an analytical test for determining structural fixed modes under arbitrary information constraint and shows its equivalence to a graph-theoretical condition.

Dealing with the original structure is also preferable since stabilizing controllers can be constructed without having to first expand their size. Finally, while the proofs in [1], (as well as [5]), are constructive in nature, they do not clearly lead to an explicit synthesis algorithm.

Many of the ideas for the rigorous proof of the necessity of the fixed mode condition were first presented in [7], while many of the ideas for the rigorous proof of its sufficiency, along with the development of the algorithm, were first presented in [8], before being refined and generalized here.

The organization of this paper is as follows. In Section II we define notation and preliminaries, including our definition of fixed modes and the controller types that we will later need. In Section III we review and establish some results for centralized controllers. In Section IV, we then state and prove our main results in two parts; Section IV-A will prove the necessity of having fixed modes in the LHP for existence of a FDLTI stabilizing controller, and Section IV-B will prove the sufficiency of the aforementioned assumption. In Section V, we give the explicit computational algorithm, along with a numerical example, followed by some concluding remarks in Section VI.

II. PRELIMINARIES

We proceed with the following preliminary definitions. Let $\Re(\cdot)$ denote the real part of any complex number. Define \mathbb{C} to be the complex plane, $\mathbb{C}^- \triangleq \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\}$ to be the open left-half plane, and $\mathbb{C}^+ \triangleq \mathbb{C} \setminus \mathbb{C}^- = \{\lambda \in \mathbb{C} \mid \Re(\lambda) \geq 0\}$ to be the closed right-half plane. Let \mathbf{e}_i denote the unit vector of all zeros except for the i^{th} element which is 1. Note that the dimension of \mathbf{e}_k should be clear from the context and thus

we suppress the explicit dimension of \mathbf{e}_k in the notation. For a real matrix A , define the following norm:

$$\|A\|_\infty = \max_i \left(\sum_j |A_{ij}| \right),$$

and let $B(\lambda_0, \epsilon) \triangleq \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \epsilon\}$ denote the open ϵ -ball around λ_0 .

We consider an FDLTI plant $P(\sigma)$ (where $\sigma = s, z$ depending on whether we are considering continuous or discrete-time cases; we use σ for statements that apply to both). We assume that P has n_u inputs, n_y outputs, and a state-space representation of P is given by (A_P, B_P, C_P, D_P) . All controllers under consideration in this paper will also be FDLTI.

We impose information constraints on the controller to encapsulate that each part of the controller may access certain sensor measurements, but not others. We define a set of admissible indices $\text{Adm}(\mathcal{S})$, such that $(i, j) \in \text{Adm}(\mathcal{S})$ if and only if controller i is allowed to access measurement j . The information constraint is then denoted by the constraint $K \in \mathcal{S}$, where $K_{ij} = 0$ for all $(i, j) \notin \text{Adm}(\mathcal{S})$, for all $K \in \mathcal{S}$.

We will make use of the following specific sparsity patterns:

- \mathcal{S}_c : Centralized sparsity patterns, i.e., no sparsity constraints are imposed on the controller. $\text{Adm}(\mathcal{S}) = \{(i, j) \mid \forall i, j\}$.
- \mathcal{S}_d : Diagonal sparsity patterns, i.e., $K(\sigma)$ must be zero for all off-diagonal terms (for almost all σ). $\text{Adm}(\mathcal{S}) = \{(i, i) \mid \forall i\}$.

We also define types of controllers that will help us to easily refer to whether a controller K is static, dynamic, or static for some elements but dynamic for others. We will make use of the following controller types:

- \mathcal{T}^d : Set of finite order dynamic controllers, i.e., A_K, B_K, C_K, D_K each are real matrices of compatible dimension.
- \mathcal{T}^s : Set of static controllers, i.e., A_K, B_K , and C_K are all zero and only D_K could be non-zero.
- $\mathcal{T}_{i,j}^{s+1}$: Set of controllers such that all of the elements of the controller are static except for the $(i, j)^{\text{th}}$ element which could be dynamic; i.e., for all $(k, l) \neq (i, j)$, we have $K_{kl} \in \mathbb{R}$, while K_{ij} may be a proper transfer function in σ . This could be read as “static plus one”.
- \mathcal{T}_I^{s+k} : Set of controllers such that all the elements of controller are static except for k indices in the set $I \triangleq \{(i_1, j_1), \dots, (i_k, j_k)\}$; i.e., for all $(k, l) \notin I$, we have $K_{kl} \in \mathbb{R}$, while K_{ij} is a proper transfer function in σ for all $(i, j) \in I$. This could be read as “static plus k ”.

For any information structure \mathcal{S} , let $a \triangleq |\text{Adm}(\mathcal{S})|$ be the number of admissible non-zero indices of the controller, and let the tuple $I \triangleq \{(i_1, j_1), \dots, (i_a, j_a)\}$ be any arbitrary ordering of these admissible non-zero indices. For any $D \in \mathcal{T}^s \cap \mathcal{S}$, we define the sequence of matrices $D|_{(m)} \in \mathbb{R}^{n_u \times n_y}$, $m \in \{0, 1, \dots, a\}$ as:

$$D|_{(0)} \triangleq 0, \quad D|_{(m)} \triangleq \sum_{l=1}^m \mathbf{e}_{i_l} D_{i_l j_l} \mathbf{e}_{j_l}^T \quad \text{for } m \in \{1, \dots, a\} \quad (1)$$

where $\mathbf{e}_{i_l} \in \mathbb{R}^{n_u}$ and $\mathbf{e}_{j_l} \in \mathbb{R}^{n_y}$, for $l \in \{1, \dots, a\}$. This $D|_{(m)}$ gives the static controller matrix with only the first m admissible indices.

The closed-loop has a state-space representation with dynamics matrix denoted by $A_{CL}(P, K)$, given by:

$$A_{CL}(P, K) \triangleq \begin{pmatrix} A_P + B_P M D_K C_P & B_P M C_K \\ B_K N C_P & A_K + B_K D_P M C_K \end{pmatrix}, \quad (2)$$

where $M \triangleq (I - D_K D_P)^{-1}$ and $N \triangleq (I - D_P D_K)^{-1}$. We have $M D_K = D_K N$ and similarly $D_P M = N D_P$, as well as $N = I + D_P M D_K$.

As illustrated in Figure 1, let $\Gamma(P, K)$ denote the map from the reference inputs to the outputs of P (i.e., from r to y), when K is closed around P . A state-space representation for

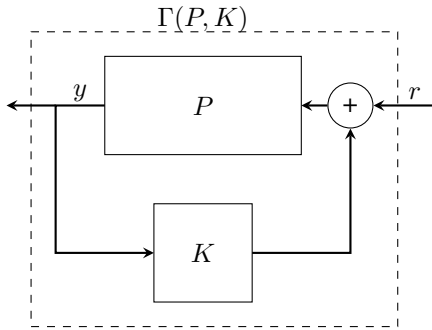


Fig. 1. The map from reference inputs to outputs when K is closed around P .

$\Gamma(P, K)$ is given by:

$$\Gamma(P, K) = \left[\begin{array}{c|c} A_{CL}(P, K) & \begin{matrix} B_P M \\ B_K D_P M \end{matrix} \\ \hline \begin{matrix} N C_P \\ D_P M C_K \end{matrix} & D_P M \end{array} \right]. \quad (3)$$

We have the following property of $\Gamma(\cdot, \cdot)$:

$$\Gamma(\Gamma(P, K_1), K_2) = \Gamma(P, K_1 + K_2), \quad (4)$$

which can be verified by working out the state-space representation of both sides.

Definition 1: The set of fixed modes of a plant P with respect to a sparsity pattern \mathcal{S} and a type \mathcal{T} , is defined to be:

$$\begin{aligned} \Lambda(P, \mathcal{S}, \mathcal{T}) &\triangleq \{\lambda \in \mathbb{C} \mid \lambda \in \text{eig}(A_{CL}(P, K)) \ \forall \ K \in \mathcal{S} \cap \mathcal{T}\} \\ &= \bigcap_{K \in \mathcal{S} \cap \mathcal{T}} \text{eig}(A_{CL}(P, K)). \end{aligned}$$

Remark 2: This reduces to the definition of fixed modes in [1] if $\mathcal{S} = \mathcal{S}_d$ and $\mathcal{T} = \mathcal{T}^s$.

For any FDLTI system P , denote its open-loop modes by $\zeta(P) = \text{eig}(A_P)$, and for each mode $\lambda \in \zeta(P)$, let $\mu(\lambda, P)$ denote its multiplicity. We can partition the open-loop modes as:

$$\zeta(P) = \Lambda(P, \mathcal{S}, \mathcal{T}^s) \cup \Lambda^\sim(P, \mathcal{S}, \mathcal{T}^s) \quad (5)$$

where

$$\Lambda^\sim(P, \mathcal{S}, \mathcal{T}^s) = \text{eig}(A_P) \setminus \Lambda(P, \mathcal{S}, \mathcal{T}^s)$$

gives the non-fixed modes, which we then further partition as:

$$\Lambda^\sim(P, \mathcal{S}, \mathcal{T}^s) = \Lambda_+^\sim(P, \mathcal{S}, \mathcal{T}^s) \cup \Lambda_-^\sim(P, \mathcal{S}, \mathcal{T}^s),$$

where

$$\begin{aligned} \Lambda_+^\sim(P, \mathcal{S}, \mathcal{T}^s) &= \{\alpha \in \zeta(P) \mid \Re(\alpha) \geq 0\} \setminus \Lambda(P, \mathcal{S}, \mathcal{T}^s) \\ &= \Lambda^\sim(P, \mathcal{S}, \mathcal{T}^s) \cap \mathbb{C}^+ \\ &= \{\alpha_1, \dots, \alpha_{|\Lambda_+^\sim(P)|}\} \end{aligned}$$

are distinct unstable non-fixed open-loop eigenvalues of P , and

$$\begin{aligned} \Lambda_-^\sim(P, \mathcal{S}, \mathcal{T}^s) &= \{\beta \in \zeta(P) \mid \Re(\beta) < 0\} \setminus \Lambda(P, \mathcal{S}, \mathcal{T}^s) \\ &= \Lambda^\sim(P, \mathcal{S}, \mathcal{T}^s) \cap \mathbb{C}^- \\ &= \{\beta_1, \dots, \beta_{|\Lambda_-^\sim(P)|}\} \end{aligned}$$

are distinct stable non-fixed open-loop eigenvalues of P . We may suppress the dependence of these collections of eigenvalues on some of their arguments when clear from context.

We note that one can adopt the notion of the multiset to discriminate between copies of a mode with multiplicity greater than one. This would have some conceptual advantages, but would unnecessarily complicate some definitions and proofs, and so we maintain the use of standard sets, while tracking the multiplicities of the modes which we will want to move (the unstable non-fixed modes). This is equally acceptable, provided that a fixed and a non-fixed mode do not have the same value, which would require the non-fixed modes to be defined as something other than the complement of those which are fixed, as above (and multiset complementation could handle this aspect nicely). Even that situation could not be problematic if we are considering the complex plane as being split into an acceptable and an unacceptable region, since such an overlap would either represent an acceptable situation, or one which is fatal anyway.

Denote the total (with multiplicities) number of unstable non-fixed modes of a plant P by

$$\nu(P) \triangleq \sum_{\alpha \in \Lambda_+^\sim(P)} \mu(\alpha, P).$$

For a matrix A , we refer to the non-negative and negative eigenvalues respectively by $\text{eig}_+(A) \triangleq \text{eig}(A) \cap \mathbb{C}^+$, and $\text{eig}_-(A) \triangleq \text{eig}(A) \cap \mathbb{C}^-$. When $\text{eig}_-(\cdot)$, and $\text{eig}_+(\cdot)$ are applied on a general LTI system P , with a slight abuse of notation, we mean the negative, and non-positive eigenvalues of the dynamic matrix of that system, i.e., $\text{eig}_-(P) \triangleq \text{eig}_-(A_P)$, and $\text{eig}_+(P) \triangleq \text{eig}_+(A_P)$.

III. CENTRALIZED RESULTS

In this section we review and establish results on controllability, observability, and fixed modes for centralized control of linear time-invariant systems. We begin with Kalman canonical form with the help of the following lemma:

Lemma 3: For every FDLTI plant P , there exists a similarity transformation matrix T such that

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_P & B_P \\ C_P & D_P \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$= \left[\begin{array}{cccc|c} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{B}_2 \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ \hline \tilde{C}_1 & 0 & \tilde{C}_2 & 0 & D_P \end{array} \right]. \quad (6)$$

In the above equation we have the following correspondence between eigenvalues of \tilde{A}_{ii} and modes of P :

- \tilde{A}_{11} : controllable and observable modes of P ,
- \tilde{A}_{22} : controllable and unobservable modes of P ,
- \tilde{A}_{33} : uncontrollable and observable modes of P ,
- \tilde{A}_{44} : uncontrollable and unobservable modes of P .

Proof: See, for example, [9]. ■

In order to reduce some of the notation, we do not explicitly show the dependence of $\tilde{A}_{ij}, \tilde{B}_i, \tilde{C}_j$ on A_P, B_P, C_P , and T , but it should be kept in mind that wherever we use Lemma 3 on a system, the resulting (\cdot) variables are function of that system's state-space matrices with its respective Kalman similarity transformation matrix.

The following lemma is useful in connecting centralized fixed modes with the familiar notion of controllability and observability. It was shown for strictly proper plants in [10]; we establish the following generalization before proceeding.

Lemma 4: Given a proper controllable and observable plant P_{co} , for almost any $D_K \in \mathcal{T}^s \cap \mathcal{S}_c$, we have that:

$$\text{eig}(A_{CL}(P_{co}, D_K)) \cap \text{eig}(P_{co}) = \emptyset. \quad (7)$$

Proof: For a strictly proper plant refer to [10, Theorem 2]. Given the proper plant P_{co} , consider the strictly proper part of it, namely $P_{co} - D$. Then, by [10, Theorem 2] the set of static feedback gains \tilde{D}_K for which $\text{eig}(A_{CL}(P_{co} - D, \tilde{D}_K)) \cap \text{eig}(P_{co} - D) \neq \emptyset$ constitute a finite union of hyperplanes in the ambient space, and hence almost any $\tilde{D}_K \in \mathcal{T}^s \cap \mathcal{S}_c$ moves the open-loop eigenvalues of $P_{co} - D$. If $(I + \tilde{D}_K D)$ is invertible, then by the change of variable $D_K = (I + \tilde{D}_K D)^{-1} \tilde{D}_K$, we have:

$$A_{CL}(P_{co} - D, \tilde{D}_K) = A_{CL}(P_{co}, D_K).$$

The proof would be finished if $(I + \tilde{D}_K D)$ is invertible for almost any \tilde{D}_K . This can be seen as $\det(I + \tilde{D}_K D) = 0$ is a non-trivial polynomial in \tilde{D}_K (choosing $\tilde{D}_K = 0$ would yield non-zero determinant), and hence the set of \tilde{D}_K for which $\det(I + \tilde{D}_K D) = 0$ is a set with dimension less than the ambient space and has zero measure. ■

Next we state the following result regarding fixed modes with respect to a centralized sparsity pattern \mathcal{S}_c , which tells us that the fixed modes of a plant with respect to a centralized information structure are precisely its uncontrollable or unobservable modes.

Lemma 5: For any FDLTI plant P ,

$$\Lambda(P, \mathcal{S}_c, \mathcal{T}^s) = \bigcup_{i=2,3,4} \text{eig}(\tilde{A}_{ii}),$$

where \tilde{A}_{ii} are the blocks in the Kalman canonical decomposition of plant P , such that the fixed modes are the union of uncontrollable or unobservable modes of P .

Proof: Denote the controllable and observable part of P by $P_{co} \triangleq \tilde{C}_1(sI - \tilde{A}_{11})^{-1} \tilde{B}_1 + D_P$. We first establish that for any arbitrary $D_K \in \mathcal{S}_c \cap \mathcal{T}^s$ that is closed around P , we have:

$$\text{eig}(A_{CL}(P, D_K)) = \text{eig}(A_{CL}(P_{co}, D_K)) \cup \left(\bigcup_{i=2,3,4} \text{eig}(\tilde{A}_{ii}) \right). \quad (8)$$

To see this, apply the similarity transformation T given in Lemma 3 on $A_{CL}(P, D_K)$. Then $TA_{CL}(P, D_K)T^{-1}$ would only differ in blocks $\tilde{A}_{11}, \tilde{A}_{21}, \tilde{A}_{13}$, and \tilde{A}_{23} compared to the open-loop \tilde{A} in (6). This leaves the structure of \tilde{A} unchanged, and renders (8).

For any $D_K \in \mathcal{S}_c \cap \mathcal{T}^s$, and for $i = 2, 3, 4$, we then have:

$$\text{eig}(\tilde{A}_{ii}) \subseteq \text{eig}(TA_{CL}(P, D_K)T^{-1}) = \text{eig}(A_{CL}(P, D_K)),$$

$$\text{and so } \bigcup_{i=2,3,4} \text{eig}(\tilde{A}_{ii}) \subseteq \Lambda(P, \mathcal{S}_c, \mathcal{T}^s).$$

For any remaining modes of P , i.e., $\lambda \in \text{eig}(\tilde{A}_{11})$, it follows from (8) and Lemma 4 that there exists a static controller $D_K \in \mathcal{S}_c \cap \mathcal{T}^s$ such that $\lambda \notin \text{eig}(A_{CL}(P, D_K))$, and so $\lambda \notin \Lambda(P, \mathcal{S}_c, \mathcal{T}^s)$. ■

Remark 6: Due to Lemma 4 and Lemma 5, almost any randomly chosen $D_K \in \mathcal{T}^s \cap \mathcal{S}_c$ moves all the open-loop modes of P , except those of $\Lambda(P, \mathcal{S}_c, \mathcal{T}^s)$.

Also to make this paper sufficiently self-contained we use our notation to restate the following result, which tells us that the fixed modes of a plant with centralized information structure are the same with respect to static or dynamic control.

Theorem 7: Given an FDLTI plant P ,

$$\Lambda(P, \mathcal{S}_c, \mathcal{T}^s) = \Lambda(P, \mathcal{S}_c, \mathcal{T}^d).$$

Proof: The \supseteq inclusion follows immediately since $\mathcal{T}^s \subseteq \mathcal{T}^d$.

We now need to show that $\Lambda(P, \mathcal{S}_c, \mathcal{T}^s) \subseteq \Lambda(P, \mathcal{S}_c, \mathcal{T}^d)$; using Lemma 5, we can achieve this by showing that $\bigcup_{i=2,3,4} \text{eig}(\tilde{A}_{ii}) \subseteq \Lambda(P, \mathcal{S}_c, \mathcal{T}^d)$, which can be achieved

by showing that $\bigcup_{i=2,3,4} \text{eig}(\tilde{A}_{ii}) \subseteq \text{eig}(A_{CL}(P, K))$ for arbitrary $K \in \mathcal{S}_c \cap \mathcal{T}^d$.

Given an arbitrary $K \in \mathcal{S}_c \cap \mathcal{T}^d$, and letting T be the similarity transformation matrix from Lemma 3, we can then apply (6) to (2) to get

$$\begin{aligned} & \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} A_{CL}(P, K) \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A} + \tilde{B}MD_K\tilde{C} & \tilde{B}MC_K \\ B_KN\tilde{C} & A_K + B_KD_PMC_K \end{pmatrix} \\ &= \left(\begin{array}{cccc|c} * & 0 & * & 0 & \tilde{B}_1MC_K \\ * & \tilde{A}_{22} & * & * & \tilde{B}_2MC_K \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & * & \tilde{A}_{44} & 0 \\ \hline B_KN\tilde{C}_1 & 0 & B_KN\tilde{C}_2 & 0 & * \end{array} \right) \end{aligned}$$

where we have let $(\tilde{A}, \tilde{B}, \tilde{C}, D_P)$ give the blocks of (6).

If we apply another similarity transformation which swaps the first/second and third/fifth row and column blocks, the result is an upper block triangular matrix for which the eigenvalues clearly include those of \tilde{A}_{22} , \tilde{A}_{33} , and \tilde{A}_{44} , as desired. ■

IV. MAIN RESULT

We will generalize the result of [1] to arbitrary information structures, and provide a comprehensive proof in this section. Section IV-A will establish the invariance of fixed modes with respect to static and dynamic controllers, thereby demonstrating the necessity of having all of the fixed modes in the LHP for decentralized stabilizability, and Section IV-B will give a constructive proof for obtaining a stabilizing controller when all of the fixed modes of P are in the LHP, thereby demonstrating the sufficiency.

A. Invariance of fixed modes

We will show in this subsection that for any arbitrary sparsity pattern \mathcal{S} , the set of fixed modes with respect to static controllers is the same as the set of fixed modes with respect to dynamic controllers.

We first state a lemma which is unsurprising but which will be helpful. This lemma states that if λ is a fixed mode of a system with respect to static controllers and sparsity pattern \mathcal{S} , then after closing the loop with an arbitrary matrix $D_K \in \mathcal{S}$, if we further allow only one of the static elements of the controller to vary, then λ will remain as a fixed mode. Given any matrix $D_K \in \mathcal{S}$, and any $(i, j) \in \text{Adm}(\mathcal{S})$, define $P^+(D_K)$, as illustrated in Figure 2, as:

$$P^+(D_K) \triangleq \mathbf{e}_j^T \Gamma(P, D_K) \mathbf{e}_i = \left[\begin{array}{c|c} A_{P+} & B_{P+} \\ \hline C_{P+} & D_{P+} \end{array} \right],$$

where $A_{P+} \triangleq A_{\text{CL}}(P, D_K) = A_P + B_P M D_K C_P$, $B_{P+} \triangleq B_P M \mathbf{e}_i$, $C_{P+} \triangleq \mathbf{e}_j^T N C_P$, and $D_{P+} \triangleq \mathbf{e}_j^T D_P M \mathbf{e}_i$. We note that this notation suppresses the dependence of P^+ on the particular choice of the admissible index pair.

Lemma 8: Given any matrix $D_K \in \mathcal{S} \cap \mathcal{T}^s$, and any $(i, j) \in \text{Adm}(\mathcal{S})$, if $\lambda \in \Lambda(P, \mathcal{S}, \mathcal{T}^s)$, then $\lambda \in \Lambda(P^+(D_K), \mathcal{S}_c, \mathcal{T}^s)$, i.e., $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subseteq \Lambda(P^+(D_K), \mathcal{S}_c, \mathcal{T}^s)$.

Proof: Suppose that $\lambda \in \Lambda(P, \mathcal{S}, \mathcal{T}^s)$.

For an arbitrary real scalar V , we have:

$$\begin{aligned} A_{\text{CL}}(P^+(D_K), V) &= A_{\text{CL}}(\mathbf{e}_j^T \Gamma(P, D_K) \mathbf{e}_i, V) \\ &= A_{\text{CL}}(\Gamma(P, D_K), \mathbf{e}_i V \mathbf{e}_j^T) \\ &\stackrel{(4)}{=} A_{\text{CL}}(P, D_K + \mathbf{e}_i V \mathbf{e}_j^T) \\ &= A_{\text{CL}}(P, D_K^V), \end{aligned} \quad (9)$$

where we have defined $D_K^V \triangleq D_K + \mathbf{e}_i V \mathbf{e}_j^T$ as the static controller which is now effectively being closed around the plant. Since we clearly have $D_K^V \in \mathcal{S} \cap \mathcal{T}^s$ and since $\lambda \in \Lambda(P, \mathcal{S}, \mathcal{T}^s)$, it follows that $\lambda \in \text{eig}(A_{\text{CL}}(P^+(D_K), V))$. Since V was arbitrary, we have $\lambda \in \Lambda(P^+(D_K), \mathcal{S}_c, \mathcal{T}^s)$. ■

Next, we relate fixed modes with respect to static controllers to those where only one of the admissible elements is allowed

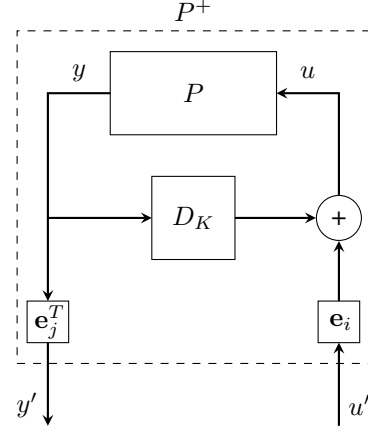


Fig. 2. P^+ is the SISO map from u' to y' .

to be dynamic; that is, to “static plus one” controllers. The lemma will prove useful because closing such a controller around the plant is equivalent to interconnecting a SISO dynamic controller with P^+ , and we can then leverage our knowledge of centralized controllers. This result will be the foundation of the induction that we want to use later on. The outline of the proof is similar to that of [1, Proposition 1].

Theorem 9: For any sparsity pattern \mathcal{S} , and any arbitrarily fixed indices $(i, j) \in \text{Adm}(\mathcal{S})$:

$$\Lambda(P, \mathcal{S}, \mathcal{T}^s) = \Lambda(P, \mathcal{S}, \mathcal{T}_{i,j}^{s+1}).$$

Proof: The \supseteq inclusion follows immediately since $\mathcal{T}^s \subseteq \mathcal{T}_{i,j}^{s+1}$.

We now need to show that $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subseteq \Lambda(P, \mathcal{S}, \mathcal{T}_{i,j}^{s+1})$. We have:

$$\begin{aligned} \Lambda(P, \mathcal{S}, \mathcal{T}^s) &\stackrel{\text{Lem.8}}{\subseteq} \bigcap_{D_K \in \mathcal{S} \cap \mathcal{T}^s} \Lambda(P^+(D_K), \mathcal{S}_c, \mathcal{T}^s) \\ &\stackrel{\text{Thm.7}}{=} \bigcap_{D_K \in \mathcal{S} \cap \mathcal{T}^s} \Lambda(P^+(D_K), \mathcal{S}_c, \mathcal{T}^d) \\ &= \bigcap_{D_K \in \mathcal{S} \cap \mathcal{T}^s} \bigcap_{k^d \in \mathcal{T}^d} \text{eig}(A_{\text{CL}}(P^+(D_K), k^d)) \\ &= \bigcap_{D_K \in \mathcal{S} \cap \mathcal{T}^s} \bigcap_{k^d \in \mathcal{T}^d} \text{eig}(\Gamma(\Gamma(P, D_K), \mathbf{e}_i k^d \mathbf{e}_j^T)) \\ &\stackrel{(4)}{=} \bigcap_{D_K \in \mathcal{S} \cap \mathcal{T}^s} \bigcap_{k^d \in \mathcal{T}^d} \text{eig}(\Gamma(P, D_K + \mathbf{e}_i k^d \mathbf{e}_j^T)) \\ &= \bigcap_{K^{s+1} \in \mathcal{S} \cap \mathcal{T}_{i,j}^{s+1}} \text{eig}(\Gamma(P, K^{s+1})) \\ &= \Lambda(P, \mathcal{S}, \mathcal{T}_{i,j}^{s+1}) \end{aligned}$$

where the penultimate equality follows since $(\mathcal{S} \cap \mathcal{T}^s) + \mathbf{e}_i \mathcal{T}^d \mathbf{e}_j^T = \mathcal{S} \cap \mathcal{T}_{i,j}^{s+1}$, and this completes the proof. ■

We note that it was this result, showing that modes which are fixed with respect to static controllers are still fixed with respect to “static plus one” controllers, that was established for $\mathcal{S} = \mathcal{S}_d$ in [1], and at which point Theorem 13 was claimed to hold true. We will now show how to extend this result to show that modes which are fixed with respect to controllers

with any given number of dynamic indices; that is, with respect to “static plus k ” controllers, are still fixed when an additional index is allowed to become dynamic; that is, with respect to “static plus $k + 1$ ” controllers. The main result will indeed follow once that has been established.

We will proceed with the following definitions. Let $K^{(k)}(\sigma)$ be the controller after k steps, with k of its indices allowed to be dynamic, and define $I^{(k)} \triangleq \{(i_1, j_1), \dots, (i_k, j_k)\} \subset \text{Adm}(\mathcal{S})$ as the set of such indices where $K^{(k)}(\sigma)$ is allowed to be dynamic, such that $K^{(k)} \in \mathcal{T}_{I^{(k)}}^{s+k} \cap \mathcal{S}$. Also let $(A_K^{(k)}, B_K^{(k)}, C_K^{(k)}, D_K^{(k)})$ be a state-space representation for $K^{(k)}(\sigma)$.

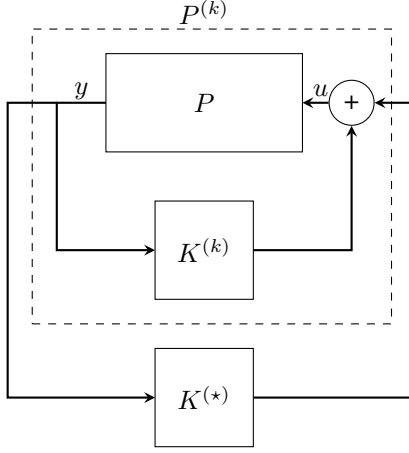


Fig. 3. Plant $P^{(k)}$ and its respective controller $K^{(*)}$.

Define $P^{(k)}(\sigma)$, illustrated in Figure 3, by closing $K^{(k)}(\sigma)$ around $P(\sigma)$ in such a way that the outputs of $P^{(k)}$ are the same as the outputs of P , and such that the inputs of $P^{(k)}$ are added to the outputs of $K^{(k)}$ and fed into P .

A state-space representation for $P^{(k)}(\sigma)$ is given by

$$P^{(k)} \triangleq \Gamma(P, K^{(k)}), \quad (10)$$

i.e., by replacing (A_K, B_K, C_K, D_K, M) with $(A_{K^{(k)}}, B_{K^{(k)}}, C_{K^{(k)}}, D_{K^{(k)}}, M^{(k)})$ in (3), where $M^{(k)} = (I - D_{K^{(k)}} D_P)^{-1}$.

We prove one remaining lemma before our main inductive step. This lemma relates the modes which are fixed when closing controllers with $k + 1$ dynamic elements around the plant, to the modes which are fixed when first closing controllers with k dynamic elements around the plant, and then closing a controller with an additional dynamic element around the resulting plant, as in Figure 3. This will allow us to use our result relating static and “static plus one” controllers to make conclusions relating “static plus k ” and “static plus $k + 1$ ” controllers.

Remark 10: We used the fact that $(\mathcal{S} \cap \mathcal{T}^s) + \mathbf{e}_i \mathcal{T}^d \mathbf{e}_j^T = \mathcal{S} \cap \mathcal{T}_{i,j}^{s+1}$; that is, that adding static controllers and a dynamic element is equivalent to taking all of the “static plus one” controllers, at the end of the proof of Theorem 9. If this could be extended to state that $(\mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}) + \mathcal{T}_{i,j}^{s+1} = \mathcal{S} \cap \mathcal{T}_{I^{(k)} \cup \{i,j\}}^{s+k+1}$; that is, that adding “static plus k ” controllers and “static plus one” controllers is equivalent to taking all of the “static plus

$k + 1$ ” controllers, then Theorem 12 would follow similarly and easily, and the upcoming lemma would be trivial and unnecessary. It is not clear, however, that a “static plus $k + 1$ ” controller can always be decomposed in that manner. We thus first introduce the following lemma, which states that, regardless of whether those two sets are the same, the modes which remain fixed as the controller varies over them are indeed identical.

Lemma 11: Given a set of indices $I^{(k)} \subset \text{Adm}(\mathcal{S})$, an additional index pair $(i, j) \in \text{Adm}(\mathcal{S}) \setminus I^{(k)}$, set $I^{(k+1)}$ to be $I^{(k+1)} \triangleq I^{(k)} \cup (i, j)$, and let $P^{(k)}$ be as in (10), then we have:

$$\Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k+1)}}^{s+k+1}) = \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}} \Lambda(P^{(k)}, \mathcal{S}, \mathcal{T}_{i,j}^{s+1}). \quad (11)$$

Proof: For ease of notation, when the controllers are unambiguous such that we can suppress the dependency upon them, define $A_{\text{CL}}^{\text{LHS}} = A_{\text{CL}}(P, K^{(k+1)})$ and $A_{\text{CL}}^{\text{RHS}} = A_{\text{CL}}(P^{(k)}, K^{(*)})$ to be the closed-loop dynamics matrices arising on each side of the equation for given controllers. Also let $\mathcal{K}_{\text{LHS}} \triangleq \{K^{(k+1)} | K^{(k+1)} \in \mathcal{T}_{I^{(k+1)}}^{s+k+1} \cap \mathcal{S}\}$, and $\mathcal{K}_{\text{RHS}} \triangleq \{(K^{(k)}, K^{(*)}) | K^{(k)} \in \mathcal{T}_{I^{(k)}}^{s+k} \cap \mathcal{S}, K^{(*)} \in \mathcal{T}_{i,j}^{s+1} \cap \mathcal{S}\}$ give the sets of controllers that must be considered on each side, such that the LHS can be abbreviated as $\bigcap_{\mathcal{K}_{\text{LHS}}} \text{eig}(A_{\text{CL}}^{\text{LHS}})$, and the RHS can be abbreviated as $\bigcap_{\mathcal{K}_{\text{RHS}}} \text{eig}(A_{\text{CL}}^{\text{RHS}})$.

First we prove the \subseteq part by showing that for every admissible $K^{(*)}(\sigma)$, i.e., $K^{(*)} \in \mathcal{S} \cap \mathcal{T}_{i,j}^{s+1}$, and admissible $K^{(k)}(\sigma)$ in RHS, there exist a $K^{(k+1)}(\sigma)$ in LHS such that $A_{\text{CL}}^{\text{RHS}} = A_{\text{CL}}^{\text{LHS}}$. To see this observe that:

$$\Gamma(\Gamma(P, K^{(k)}), K^{(*)}) \stackrel{(4)}{=} \Gamma(P, K^{(k)} + K^{(*)}).$$

Thus we choose $K^{(k+1)}(\sigma) = K^{(k)}(\sigma) + K^{(*)}(\sigma)$. This $K^{(k+1)}$ is admissible because it has only one further dynamic element at position $(i, j) \in \text{Adm}(\mathcal{S})$, and thus is in $\mathcal{T}_{I^{(k+1)}}^{s+k+1}$. Hence for every admissible $(K^{(k)}, K^{(*)})$, there exists an admissible $K^{(k+1)} \in \mathcal{K}_{\text{LHS}}$ constructed as above such that $A_{\text{CL}}^{\text{LHS}} = A_{\text{CL}}^{\text{RHS}}$, and so $\bigcap_{\mathcal{K}_{\text{LHS}}} \text{eig}(A_{\text{CL}}^{\text{LHS}}) \subseteq \bigcap_{\mathcal{K}_{\text{RHS}}} \text{eig}(A_{\text{CL}}^{\text{RHS}})$.

We will prove the \supseteq part by contradiction. Assume that (11) does not hold, and thus that there exists a λ such that $\lambda \in \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}} \Lambda(P^{(k)}, \mathcal{S}, \mathcal{T}_{i,j}^{s+1})$, but $\lambda \notin \Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k+1)}}^{s+k+1})$. Then we have:

$$\lambda \in \text{eig}(A_{\text{CL}}(P^{(k)}, K^{(*)})) \quad \forall (K^{(k)}, K^{(*)}) \in \mathcal{K}_{\text{RHS}}, \quad (12a)$$

$$\exists K^{(k+1)} \in \mathcal{K}_{\text{LHS}} \text{ s.t. } \lambda \notin \text{eig}(A_{\text{CL}}(P, K^{(k+1)})). \quad (12b)$$

Starting with $K^{(k+1)}$ from (12b), we will show that we can then construct a $K^{(k)}$ and $K^{(*)}$ to falsify (12a).

Based on $K^{(k+1)}$ in (12b), we let $\tilde{K}^{(*)}$ be the dynamic part of the final dynamic index by defining $\tilde{K}^{(*)} =$

$(\tilde{A}_K^{(*)}, \tilde{B}_K^{(*)}, \tilde{C}_K^{(*)}, \tilde{D}_K^{(*)})$ as:

$$\begin{aligned}\tilde{A}_K^{(*)} &= A_K^{(k+1)}, \\ \tilde{B}_K^{(*)} &= B_K^{(k+1)} \mathbf{e}_j \mathbf{e}_j^T \\ &= \begin{bmatrix} 0 & \cdots & B_{K,j}^{(k+1)} & \cdots & 0 \end{bmatrix}, \\ \tilde{C}_K^{(*)} &= \mathbf{e}_i \mathbf{e}_i^T C_K^{(k+1)} \\ &= \begin{bmatrix} 0 & \cdots & (C_{K,i}^{(k+1)})^T & \cdots & 0 \end{bmatrix}^T, \\ \tilde{D}_K^{(*)} &= 0,\end{aligned}$$

i.e., $\tilde{B}_K^{(*)}$ is of the same dimension as $B_K^{(k+1)}$ with all its columns being zero except the j -th column, and $\tilde{C}_K^{(*)}$ is of the same dimension as $C_K^{(k+1)}$ with all of its rows being zero except the i -th row. Then define $\tilde{K}^{(k)} \triangleq K^{(k+1)} - \tilde{K}^{(*)}$, thus a state-space representation for $\tilde{K}^{(k)}$ is:

$$\begin{aligned}\tilde{A}_K^{(k)} &= \text{diag}(A_K^{(k+1)}, A_K^{(k+1)}), \\ \tilde{B}_K^{(k)} &= \begin{bmatrix} (B_K^{(k+1)})^T & (\tilde{B}_K^{(*)})^T \end{bmatrix}^T, \\ \tilde{C}_K^{(k)} &= \begin{bmatrix} C_K^{(k+1)} & -\tilde{C}_K^{(*)} \end{bmatrix}, \\ \tilde{D}_K^{(k)} &= D_K^{(k+1)}.\end{aligned}$$

Construct $\tilde{P}^{(k)}$ in the same way as illustrated in Figure 3 by closing $\tilde{K}^{(k)}$ around P . Now if we use the following similarity transformation T on $A_{\text{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(*)})$,

$$T = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & I \end{bmatrix},$$

then $TA_{\text{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(*)})T^{-1}$ results in an upper block triangular matrix with blocks $A_K^{(k+1)}$, $A_{\text{CL}}(P, K^{(k+1)})$, and $A_K^{(k+1)}$, indicating that:

$$\begin{aligned}\text{eig}\left(A_{\text{CL}}(\tilde{P}^{(k)}, \tilde{K}^{(*)})\right) &= \\ \text{eig}\left(A_{\text{CL}}(P, K^{(k+1)})\right) \cup \text{eig}\left(A_K^{(k+1)}\right).\end{aligned}\quad (13)$$

Thus (12) can only be satisfied if:

$$\lambda \in \text{eig}\left(A_K^{(k+1)}\right). \quad (14)$$

We have shown that the only way to have an eigenvalue which is not on the LHS (when $K^{(k+1)}$ is closed around the plant) but which is on the RHS (when $\tilde{K}^{(*)}$ and $\tilde{K}^{(k)}$ are then constructed as above), is if it comes from the dynamics matrix of $K^{(k+1)}$. We will now finish the proof by showing that if this is the case, we can make a small perturbation to the matrix such that it no longer has this eigenvalue, thus removing it from the RHS, while it is still not a closed-loop eigenvalue on the LHS.

Construct $\hat{K}^{(k+1)}$ by perturbing the A matrix of $K^{(k+1)}$;

that is, $\hat{K}^{(k+1)}$ is defined by:

$$\begin{aligned}\hat{A}_K^{(k+1)} &= \tilde{A}_K^{(k+1)} + \epsilon I, & \hat{B}_K^{(k+1)} &= \tilde{B}_K^{(k+1)}, \\ \hat{C}_K^{(k+1)} &= \tilde{C}_K^{(k+1)}, & \hat{D}_K^{(k+1)} &= \tilde{D}_K^{(k+1)}.\end{aligned}$$

For sufficiently small ϵ this yields

$$\lambda \notin \text{eig}\left(\hat{A}_K^{(k+1)}\right). \quad (15)$$

Using the same steps as before to construct $\hat{K}^{(*)}$ and $\hat{K}^{(k)}$ results in ϵI also being added to $\tilde{A}_K^{(*)}$ and $\tilde{A}_K^{(k)}$. Then using the same similarity transformation T used to derive (13), we have

$$\begin{aligned}\text{eig}\left(A_{\text{CL}}(\hat{P}^{(k)}, \hat{K}^{(*)})\right) &= \\ \text{eig}\left(A_{\text{CL}}(P, \hat{K}^{(k+1)})\right) \cup \text{eig}\left(\hat{A}_K^{(k+1)}\right),\end{aligned}\quad (16)$$

where $\hat{P}^{(k)}$ is constructed by closing $\hat{K}^{(k)}$ around P , as illustrated for the unperturbed systems in Figure 3.

Since $A_{\text{CL}}(P, K^{(k+1)})$ is continuous in the entries of $K^{(k+1)}$, and since the eigenvalues of a matrix are continuous in its entries (see, for example [11, Theorem 5.2. on p. 89]), it follows that by a sufficiently small perturbation made to $K^{(k+1)}$, along with (12b), we still have $\lambda \notin \text{eig}\left(A_{\text{CL}}(P, \hat{K}^{(k+1)})\right)$. It then follows from (15) and (16) that $\lambda \notin \text{eig}\left(A_{\text{CL}}(\hat{P}^{(k)}, \hat{K}^{(*)})\right)$.

Thus we have been able to show that there exists a $(\hat{K}^{(k)}, \hat{K}^{(*)}) \in \mathcal{K}_{\text{RHS}}$ such that $\lambda \notin \text{eig}\left(A_{\text{CL}}(\hat{P}^{(k)}, \hat{K}^{(*)})\right)$, which contradicts our assumption. ■

Now we are ready to prove our main inductive step: that given a certain number of controller indices which are allowed to be dynamic, and the associated set of fixed modes, allowing one additional index to become dynamic does not change the fixed modes.

Theorem 12: Given an FDLTI plant P , a sparsity pattern \mathcal{S} , an admissible set of dynamic elements at step k denoted by $I^{(k)} \subset \text{Adm}(\mathcal{S})$, an index pair $(i, j) \in \text{Adm}(\mathcal{S}) \setminus I^{(k)}$ that is further allowed to be dynamic at step $k+1$, and the resulting $I^{(k+1)} = I^{(k)} \cup (i, j)$, we have:

$$\Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k)}}^{s+k}) = \Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k+1)}}^{s+k+1}).$$

Proof: Beginning with the quantity on the right-hand side, we get:

$$\begin{aligned}\Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k+1)}}^{s+k+1}) &\stackrel{\text{Lem.11}}{=} \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}} \Lambda(P^{(k)}, \mathcal{S}, \mathcal{T}_{i,j}^{s+1}) \\ &\stackrel{\text{Thm.9}}{=} \bigcap_{K^{(k)} \in \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}} \Lambda(P^{(k)}, \mathcal{S}, \mathcal{T}^s) \\ &= \Lambda(P, \mathcal{S}, \mathcal{T}_{I^{(k)}}^{s+k}),\end{aligned}$$

where the final equality follows since clearly $(\mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}) + (\mathcal{S} \cap \mathcal{T}^s) = \mathcal{S} \cap \mathcal{T}_{I^{(k)}}^{s+k}$, and this completes the proof. ■

We can now state and easily prove our main result. The

following shows that for any FDLTI plant P , and any sparsity pattern \mathcal{S} , the set of fixed modes with respect to static and dynamic controllers are the same.

Theorem 13: Given plant P , and sparsity constraint \mathcal{S} :

$$\Lambda(P, \mathcal{S}, \mathcal{T}^s) = \Lambda(P, \mathcal{S}, \mathcal{T}^d). \quad (17)$$

Proof: This follows by induction from Theorem 12. ■

B. Stabilization

The results from the previous subsection tell us that having all of the fixed modes in the LHP is necessary for stabilizability with respect to FDLTI controllers with the given structure. We now address the sufficiency of the condition. With a constructive proof, we will show that we can stabilize a plant P with arbitrary information structure \mathcal{S} , as long as it has no unstable fixed modes. We will achieve this by showing that we can always find a controller which will reduce the number of unstable modes, while leaving all of the fixed modes in the LHP, which can then be applied as many times as required.

We will first state the following lemma from [1], which gives some properties regarding continuity and topology of non-fixed modes with respect to static controllers. It tells us that we can keep the modes within a given distance of the original ones by closing a small enough matrix D around the plant, and that an arbitrarily small D can move all of the non-fixed modes.

Lemma 14: For any plant P , and any information structure \mathcal{S} , partition the open-loop eigenvalues of P as in (5), then we have:

- 1) For all $\epsilon > 0$, there exist $\gamma > 0$ such that for all $D \in \mathcal{S} \cap \mathcal{T}^s$ with $\|D\|_\infty < \gamma$, there are exactly $\mu(\lambda, P)$ eigenvalues of $A_{CL}(P, D)$ in $B(\lambda, \epsilon)$, for all $\lambda \in \Lambda^\sim(P)$.
- 2) For all $\gamma > 0$, there exist $D \in \mathcal{T}^s \cap \mathcal{S}$ with $\|D\|_\infty < \gamma$, such that $\lambda \notin \text{eig}(A_{CL}(P, D))$, for all $\lambda \in \Lambda^\sim(P)$.

Proof: See Lemma 4 in [1]. The proof was developed for strictly proper plants with diagonal information structure, however, it does not use any property specific to only block-diagonal information structure and thus could be replaced by any arbitrary information structure. To generalize it for the proper plants, a similar change of variable technique as in proof of Lemma 4 can be used, which would add an invertibility constraint that almost always holds. ■

Remark 15: It follows from the proof that the set of D which violate part 2 of Lemma 14, forms a subset with zero Lebesgue measure, and thus a random $D \in \mathcal{S}$ that is sufficiently small satisfies all of the conditions of Lemma 14. Precisely, the space of static controllers that does not move the non-fixed modes is constructed by a finite union of hyper-surfaces in $(\mathcal{T}^s \cap \mathcal{S}) \subset \mathbb{R}^{n_u \times n_y}$. Thus, a D that satisfies all of the conditions of Lemma 14, can be found with probability one by randomly choosing the direction of $D \in \mathcal{T}^s \cap \mathcal{S}$, and then scaling it appropriately such that $\|D\|_\infty < \gamma$.

We now establish the following theorem, which shows how a given non-fixed mode can be extracted as a controllable and observable mode of a specific SISO system, as illustrated in Figure 4.

Theorem 16: For any plant P with $|\Lambda_+^\sim(P)| \geq 1$, and all fixed modes in the LHP (i.e., $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$), there exists a $D_K \in \mathcal{T}^s \cap \mathcal{S}$, and an integer $m \in \{1, \dots, a\}$, such that if we define the following SISO system:

$$\begin{aligned} P_m &= \mathbf{e}_{j_m}^T \Gamma(P, D_K) \mathbf{e}_{i_m} = \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} \\ &\triangleq \begin{bmatrix} A_P + B_P M D_K C_P & B_P M \mathbf{e}_{i_m} \\ \mathbf{e}_{j_m}^T N C_P & \mathbf{e}_{j_m}^T D_P M \mathbf{e}_{i_m} \end{bmatrix}, \end{aligned} \quad (18)$$

the following then hold:

- 1) There exists $\alpha \in \Lambda_+^\sim(P)$, such that α is a controllable and observable mode of P_m ;
- 2) The total number of unstable modes of P_m is no greater than that of P , i.e., $\nu(P_m) \leq \nu(P)$.

Proof: The outline of the proof is as follows. We will first find a $D \in \mathcal{S} \cap \mathcal{T}^s$ that when closed around P , will move all of its non-fixed modes, and will identify the index $m \in \{1, \dots, a\}$ for which $D|_{(m)}$ is the first in the sequence to alter all of them. This means that only changing the $(i_m, j_m)^{\text{th}}$ element of the static controller will change unstable mode(s) of the closed-loop, and thus those modes must be in the controllable and observable modes of the SISO plant from u_{i_m} to y_{j_m} .

Proof of part 1: Since $\Lambda_+^\sim(P) \subseteq \Lambda^\sim(P)$, Lemma 14 guarantees that we can take the static gain $D \in \mathcal{S} \cap \mathcal{T}^s$ such that when closed around P , would move all of its unstable non-fixed modes. It also asserts that by choosing this D small enough, the closed loop $A_{CL}(P, D)$ would have no more unstable modes than P itself.

Construct a sequence of matrices $D|_{(m)} \in \mathcal{T}^s \cap \mathcal{S}$ as in (1), so that $D|_{(a)} = D$ and $D|_{(0)} = 0$, thus:

$$\begin{aligned} \forall \alpha \in \Lambda_+^\sim(P) : & \quad \alpha \notin \text{eig}(A_{CL}(P, D|_{(a)})), \\ \forall \alpha \in \Lambda_+^\sim(P) : & \quad \alpha \in \text{eig}(A_{CL}(P, D|_{(0)})). \end{aligned}$$

By decreasing m from a to 1, there must exist a value of $m \in \{1, \dots, a\}$, such that:

$$\begin{aligned} \forall \alpha \in \Lambda_+^\sim(P) : & \quad \alpha \notin \text{eig}(A_{CL}(P, D|_{(m)})), \quad (19a) \\ \exists \alpha \in \Lambda_+^\sim(P) : & \quad \alpha \in \text{eig}(A_{CL}(P, D|_{(m-1)})); \quad (19b) \end{aligned}$$

that is, m is the first index for which all of the unstable non-fixed modes have been moved. If we then set $D_K = D|_{(m-1)}$ and use the definitions from (18), as illustrated in Figure 4, similar to (9) we have:

$$\begin{aligned} A_{CL}(P, D|_{(m)}) &= A_{CL}(P, D|_{(m-1)}) + \mathbf{e}_{i_m} D_{i_m, j_m} \mathbf{e}_{j_m}^T \\ &\stackrel{(4)}{=} A_{CL}(\Gamma(P, D|_{(m-1)}), \mathbf{e}_{i_m} D_{i_m, j_m} \mathbf{e}_{j_m}^T) \\ &= A_{CL}(\mathbf{e}_{j_m}^T \Gamma(P, D|_{(m-1)}) \mathbf{e}_{i_m}, D_{i_m, j_m}) \\ &= A_{CL}(P_m, D_{i_m, j_m}). \end{aligned} \quad (20)$$

From (19b), there exists at least one $\alpha \in \Lambda_+^\sim(P)$ such that:

$$\alpha \in \text{eig}(A_{CL}(P, D|_{(m-1)})) = \text{eig}(A_m),$$

but due to (19a),

$$\alpha \notin \text{eig}(A_{CL}(P, D|_{(m)})) \stackrel{(20)}{=} \text{eig}(A_{CL}(P_m, D_{i_m, j_m})).$$

For all such α that are thus moved by only closing D_{i_m, j_m}

around the SISO system P_m (for which the only information structure is the centralized one, \mathcal{S}_c), we have:

$$\begin{aligned} \exists D_{i_m, j_m} \in \mathbb{R} \text{ s.t. : } \alpha \notin \text{eig}(A_{\text{CL}}(P_m, D_{i_m, j_m})) \\ \Rightarrow \alpha \notin \Lambda(P_m, \mathcal{S}_c, \mathcal{T}^s). \end{aligned}$$

Finally, due to Lemma 5, the fixed modes of any FDLTI plant with centralized information structure are equal to its unobservable or uncontrollable modes, we must have that those α are controllable and observable modes of P_m .

Proof of part 2: since $A_m = A_{\text{CL}}(P, D|_{(m-1)})$, we need to show that this $D|_{(m-1)}$ satisfies Lemma 14.1 when we take the ϵ -balls in Lemma 14 small enough such that they do not intersect with \mathbb{C}^+ . However this is the case since the given D in part 1 of the proof satisfies Lemma 14, and $D|_{(m)}$ that are constructed from this D , satisfy $\|D|_{(m)}\|_\infty \leq \|D\|_\infty \leq \gamma$ for any $m \in \{0, 1, \dots, a\}$ based on the definition. ■

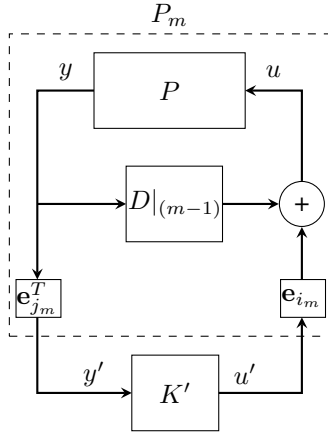


Fig. 4. P_m is the SISO map from u' to y' , and K_m is the map from y to u , giving the total control for the original plant.

In the following proposition, we will use observer-based pole placement for a centralized information structure to show how one can stabilize unstable, non-fixed modes of P_m in (18). We will add one further design constraint that the unstable modes of the controller would be different than that of P_m , and will show that this constraint is always achievable by a small perturbation of the gains. This ensures that an induction-based argument can be used later on. This constraint is not mentioned in [1], and it is unclear that without such a constraint how one can guarantee that a rigorous induction could follow, even for a diagonal information structure.

Proposition 17: All of the controllable and observable unstable modes of the plant P_m can be stabilized by an observer-based controller K' such that:

$$\text{eig}_+(K') \cap \text{eig}_+(\Gamma(P_m, K')) = \emptyset. \quad (21)$$

Proof: Our proof is in a constructive manner, we will first find a K' to only stabilize the controllable and observable modes of P_m without considering (21). We will then show that (21) is not satisfied only on a set with zero measure, and thus almost any small perturbation in the specific elements of K' will satisfy (21).

First find a similarity transformation T that will put P_m in its Kalman canonical form, therefore we would have:

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & D_m \end{array} \right] \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11}^m & 0 & \tilde{A}_{13}^m & 0 & \tilde{B}_1^m \\ \tilde{A}_{21}^m & \tilde{A}_{22}^m & \tilde{A}_{23}^m & \tilde{A}_{24}^m & \tilde{B}_2^m \\ 0 & 0 & \tilde{A}_{33}^m & 0 & 0 \\ 0 & 0 & \tilde{A}_{43}^m & \tilde{A}_{44}^m & 0 \\ \hline \tilde{C}_1^m & 0 & \tilde{C}_2^m & 0 & D_m \end{bmatrix}, \quad (22)$$

where as before, all the $(\tilde{\cdot})$ parameters depend on the transformation matrix T and the state-space representation of P_m . We want to stabilize all the unstable modes in \tilde{A}_{11} . Since based on definition $(\tilde{A}_{11}, \tilde{B}_1)$ is a controllable pair and $(\tilde{A}_{11}, \tilde{C}_1)$ is an observable pair, there exists a state feedback gain F and an observer gain L , such that eigenvalues of $\tilde{A}_{11} - \tilde{B}_1 F$ and $\tilde{A}_{11} - L\tilde{C}_1$ can be arbitrary assigned, and hence can be stabilized. We will now show that the following controller will stabilize all the unstable modes of \tilde{A}_{11} . Take the controller as:

$$K' = \left[\begin{array}{c|c} A' & B' \\ \hline C' & 0 \end{array} \right] = \left[\begin{array}{c|c} \tilde{A}_{11} - \tilde{B}_1 F - L\tilde{C}_1 + L D_m F & L \\ \hline -F & 0 \end{array} \right];$$

apply T from (22) on P_m and close K' around it, then the closed-loop $A_{\text{CL}}(P_m, K')$ would be:

$$\begin{pmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} & 0 & -\tilde{B}_1 F \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & -\tilde{B}_2 F \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ L\tilde{C}_1 & 0 & L\tilde{C}_2 & 0 & \tilde{A}_{11} - \tilde{B}_1 F - L\tilde{C}_1 \end{pmatrix}$$

Apply another similarity transformation T_1 , which keeps the first four rows the same and subtract the first row from the fifth, then we have:

$$\text{eig}(A_{\text{CL}}(P_m, K')) = \text{eig}(T_1 A_{\text{CL}}(P_m, K') T_1^{-1}) = \text{eig} \begin{pmatrix} \tilde{A}_{11} - \tilde{B}_1 F & 0 & \tilde{A}_{13} & 0 & -\tilde{B}_1 F \\ \tilde{A}_{21} - \tilde{B}_2 F & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & -\tilde{B}_2 F \\ 0 & 0 & \tilde{A}_{33} & 0 & 0 \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} & 0 \\ 0 & 0 & L\tilde{C}_2 - \tilde{A}_{13} & 0 & \tilde{A}_{11} - L\tilde{C}_1 \end{pmatrix}$$

Thus the eigenvalue of the closed loop would be

$$\text{eig}(A_{\text{CL}}(P_m, K')) = \text{eig}(\tilde{A}_{11} - \tilde{B}_1 F) \cup \text{eig}(\tilde{A}_{11} - L\tilde{C}_1) \cup \left(\bigcup_{i=2}^4 \text{eig}(\tilde{A}_{ii}) \right)$$

Therefore for all observer-based controllers that naturally satisfy $\text{eig}(\tilde{A}_{11} - \tilde{B}_1 F) \in \mathbb{C}^-$ and $\text{eig}(\tilde{A}_{11} - L\tilde{C}_1) \in \mathbb{C}^-$; unstable modes of $\Gamma(P_m, K')$ would be independent of F and L , i.e.:

$$\text{eig}_+(\Gamma(P_m, K')) = \bigcup_{i=2}^4 \text{eig}_+(\tilde{A}_{ii}), \quad (23)$$

and all unstable modes in \tilde{A}_{11} can be stabilized by appropriate choice of matrices F and L .

We will now show that (21) is not met on a set with zero

measure in the ambient space of L . Replacing (23) in (21) yield that constraint (21) is met if and only if:

$$\text{eig}_+(K') \cap \left(\bigcup_{i=2}^4 \text{eig}_+(\tilde{A}_{ii}) \right) = \emptyset, \quad (24)$$

and if not, we enforce (21) by appropriately perturbing the L matrix. Construct the perturbed controller \hat{K}' by replacing L in K' with $\hat{L} = L + L_\epsilon$, i.e.:

$$\hat{K}' \triangleq \left[\begin{array}{c|c} \hat{A}' & \hat{L} \\ \hline -F & 0 \end{array} \right],$$

with $\hat{A}' \triangleq \tilde{A}_{11} - \tilde{B}_1 F - \hat{L} \tilde{C}_1 + \hat{L} D_m F$. We want to show that \hat{K}' satisfies (24) for almost any L_ϵ . To see this, first define W as:

$$W \triangleq \left[\begin{array}{c|c} A' & I \\ \hline -\tilde{C}_1 + D_m F & 0 \end{array} \right].$$

It is also straightforward to verify that $A_{\text{CL}}(W, L_\epsilon) = \hat{A}'$. We want to apply Remark 6 on W to show that almost any perturbation L_ϵ moves all the unstable open-loop modes of W (which is equivalent to the unstable modes of K' as $\text{eig}(W) = \text{eig}(K')$). This would be achieved by showing that non of the unstable modes of W would be a fixed one, precisely:

$$\begin{aligned} A_{\text{CL}}(W, -L) &= \tilde{A}_{11} - \tilde{B}_1 F \\ \Rightarrow \Lambda(W, \mathcal{S}_c, \mathcal{T}^s) &\subseteq \text{eig}(\tilde{A}_{11} - \tilde{B}_1 F) \subset \mathbb{C}^-, \end{aligned}$$

as F is chosen to stabilize \tilde{A}_{11} . Moreover, given that $\text{eig}(\tilde{A}_{11} - \tilde{L} \tilde{C}_1) \subset \mathbb{C}^-$, if we chose L_ϵ sufficiently small, then due to a continuity argument we have $\text{eig}(\tilde{A}_{11} - \hat{L} \tilde{C}_1) \subset \mathbb{C}^-$. Thus any sufficiently small perturbation L_ϵ will make \hat{K}' satisfy (21) while still keeping $\tilde{A}_{11} - \hat{L} \tilde{C}_1$ stable. ■

We will encapsulate the desired properties of the intermediate controller at each step that partially stabilizes the plant in the following corollary, which combines Theorem 16 and Proposition 17.

Corollary 18: For every plant P that satisfies the assumptions of Theorem 16, there exists an $m \in \{1, \dots, a\}$ and a controller $K_m \in \mathcal{S} \cap \mathcal{T}_{i_m, j_m}^{s+1}$ such that:

$$\nu(\Gamma(P, K_m)) \leq \nu(P) - 1, \quad (25)$$

$$\text{eig}_+(K_m) \cap \Lambda_+^\sim(\Gamma(P, K_m)) = \emptyset. \quad (26)$$

Proof: Use Theorem 16 to find D_K and m , use Proposition 17 to find K' , and construct the MIMO controller $K_m \triangleq D|_{(m-1)} + \mathbf{e}_{i_m} K' \mathbf{e}_{j_m}^T$. As illustrated in Figure 4, this K_m has the following state-space representation:

$$K_m = \left[\begin{array}{c|c} A_m^K & B_m^K \\ \hline C_m^K & D_m^K \end{array} \right] = \left[\begin{array}{c|c} A' & B' \mathbf{e}_{j_m}^T \\ \hline \mathbf{e}_{i_m} C' & D_K \end{array} \right], \quad (27)$$

and clearly satisfies:

$$A_{\text{CL}}(P_m, K') = A_{\text{CL}}(P, K_m). \quad (28)$$

Due to Theorem 16 and Proposition 17, K' will stabilize at least one unstable mode of P , hence we have $\nu(\Gamma(P_m, K')) \leq$

$\nu(P) - 1$, and thus (25) would be an immediate result of this property of K' combined with (28). Finally, (26) follows from (21) as $A_m^K = A'$ and $\Lambda_+^\sim(\Gamma(P_m, K')) = \Lambda_+^\sim(\Gamma(P, K_m))$, due to (27) and (28). ■

We use induction to prove that if all the fixed modes of P are in LHP, then we can stabilize P by dynamic controller. We will first define the following interconnection that will be useful in the induction. Let $G^{(0)} \triangleq P$ and at each step k , denote the transfer function from u to y , as illustrated in Figure 5, by $G^{(k+1)}$, i.e., $G^{(k+1)} = \Gamma(G^{(k)}, K_m^{(k)})$. Let $(A_G^{(k)}, B_G^{(k)}, C_G^{(k)}, D_G^{(k)})$ be a state-space representation for $G^{(k)}$, also denote the total number of unstable modes of $G^{(k)}$ by $\nu^{(k)}$:

$$\nu^{(k)} \triangleq \sum_{\alpha \in \Lambda_+^\sim(G^{(k)})} \mu(\alpha, G^{(k)}).$$

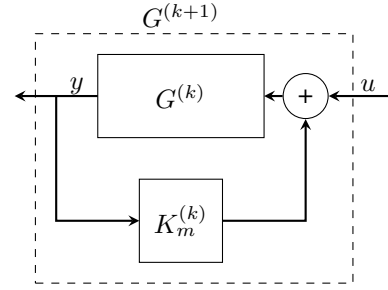


Fig. 5. Plant $G^{(k+1)} \triangleq \Gamma(G^{(k)}, K_m^{(k)})$.

The induction will be in such a way that in each step k , we will find an integer $m^{(k)} \in \{1, \dots, a\}$, and a $K_m^{(k)} \in \mathcal{T}_{i_m^{(k)}, j_m^{(k)}}^{s+1} \cap \mathcal{S}$ that when closed around $G^{(k)}$, will stabilize at least one unstable mode of $G^{(k)}$, thus $\nu^{(k+1)} \leq \nu^{(k)} - 1$. Then we will treat the corresponding $G^{(k+1)}$ as the new plant for which we want to stabilize the rest of remaining $\nu^{(k+1)}$ unstable eigenvalues, thus in at most $\nu^{(0)}$ steps, P will be stabilized. A crucial part of induction is that $G^{(k+1)}$ must have no fixed mode in closed RHP, this is not addressed in [1] and at this point it is directly claimed that Theorem 21 holds true. We will formalize this fact with the help of following lemma. It is enough to show that closing K_m around P does not add any unstable fixed modes to $\Gamma(P, K_m)$.

Lemma 19: Assume that all the fixed modes of P are in LHP, i.e.:

$$\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-, \quad (29)$$

also, assume that a controller K_m is such that it satisfies (26), then we have:

$$\Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-.$$

Proof: Proof is done by contradiction, we will first create the following set-up to state the idea. Let (A_K, B_K, C_K, D_K) be a state-space representation for K_m . We have:

$$\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subseteq \Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s),$$

since the RHS is the set of fixed modes with respect to controllers in the form $K_m + \mathcal{S} \cap \mathcal{T}^s$, whereas the LHS equals $\Lambda(P, \mathcal{S}, \mathcal{T}^d)$ (by Theorem 13), that is the set of fixed

modes with respect to controllers in $\mathcal{S} \cap \mathcal{T}^d$, which is a bigger set than $K_m \in \mathcal{S} \cap \mathcal{T}^d$. Next, it is trivial to check that if we close $-K_m$ around $\Gamma(P, K_m)$, then by applying a similarity transformation T_2 , a state-space realization that does not omit non-minimal modes of $\Gamma(\Gamma(P, K_m), -K_m)$ can be written as:

$$\begin{bmatrix} T_2 & 0 \\ 0 & I \end{bmatrix} \Gamma(\Gamma(P, K_m), -K_m) \begin{bmatrix} T_2^{-1} & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{cc|c|c} A_P & B_P C_K & 0 & B_P \\ 0 & A_K & 0 & 0 \\ \hline \bar{B}_K \bar{C}_P & \bar{B}_K \bar{D}_P \bar{C}_K & \bar{A}_K & \bar{B}_K \bar{D}_P \\ \hline C_P & D_P C_K & 0 & D_P \end{array} \right], \quad (30)$$

thus we have

$$\text{eig}(\Gamma(\Gamma(P, K_m), -K_m)) = \text{eig}(A_P) \cup \text{eig}(A_K).$$

Furthermore, due to (29), there exist a $D \in \mathcal{T}^s \cap \mathcal{S}$ that will move all the unstable modes of A_P . If we apply the same D on (30), due to the block-diagonal structure we have $\text{eig}(A_{\text{CL}}(\Gamma(\Gamma(P, K_m), -K_m), D)) = \text{eig}(A_{\text{CL}}(P, D)) \cup \text{eig}(A_K)$, which yields:

$$\Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s) \subseteq \Lambda(P, \mathcal{S}, \mathcal{T}^s) \cup \text{eig}(A_K). \quad (31)$$

Now we are ready to do the main contradiction part, assume that there exist an $\alpha \in \Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s)$, with $\Re(\alpha) \geq 0$, then

$$\begin{aligned} \alpha &\in \Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s), \quad \Re(\alpha) \geq 0 \\ &\stackrel{(31)}{\subseteq} \Lambda(P, \mathcal{S}, \mathcal{T}^s) \cup \text{eig}(A_K) \\ &\stackrel{(29)}{\Rightarrow} \alpha \in \text{eig}(A_K) \\ &\stackrel{(26)}{\Rightarrow} \alpha \notin \text{eig}(\Gamma(P, K_m)) \\ &\Rightarrow \alpha \notin \Lambda(\Gamma(P, K_m), \mathcal{S}, \mathcal{T}^s) \end{aligned}$$

thus we have achieved the desired contradiction. ■

Constraint (26) in Corollary 18 ensures that the unstable modes be non-overlapping, and is one sufficient condition to prove Lemma 19. When this condition is not met for an arbitrary choice of the feedback/observer gain, one way to always make it feasible is by adding the perturbation L_ϵ to the observer gain, which in turn might prevent exact pole placement. However, one can place the poles arbitrarily close to the desired locations by choosing L_ϵ sufficiently small.

Now we are ready to claim that if all the fixed modes of P are in the LHP, then we can stabilize P by a dynamic controller. This stabilizing controller would be a summation of individual controllers $K_m^{(k)}$, each obtained in one step of the induction, where in each step k , $K_m^{(k)}$ would only have one dynamic element (i.e., $K_m^{(k)} \in \mathcal{T}_{i_m^{(k)}, j_m^{(k)}}^{s+1} \cap \mathcal{S}$, for some $m^{(k)} \in \{1, \dots, a\}$).

Theorem 20: For any FDLTI plant P , and any information structure \mathcal{S} , if $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$, then there exist a controller $K \in \mathcal{S} \cap \mathcal{T}^d$ that will stabilize P .

Proof: Proof is done by induction. Take $k \leftarrow 0$ and let $G^{(0)} \triangleq P$. As per assumption of this theorem, $\Lambda(G^{(0)}, \mathcal{S}, \mathcal{T}^s) = \Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$. At each induction step k , we would stabilize at least one of the unstable modes of $G^{(k)}$ by Corollary 18. Specifically, with P replaced by

$G^{(k)}$ in Corollary 18, we can find a $m^{(k)} \in \{1, \dots, a\}$, and a controller $K_m^{(k)} \in \mathcal{S} \cap \mathcal{T}_{i_m^{(k)}, j_m^{(k)}}^{s+1}$, that will stabilize at least one of unstable modes of $G^{(k)}$. This $K_m^{(k)}$ satisfies (26) (with P replaced by $G^{(k)}$), and thus by Lemma 19, $G^{(k+1)} = \Gamma(G^{(k)}, K_m^{(k)})$, would have all of its fixed modes in LHP, i.e., $\Lambda(G^{(k+1)}, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$. This guarantees that we can proceed with the induction by taking $k \leftarrow k+1$, as long as $G^{(k)}$ has any remaining unstable mode. Since at each step at least one unstable mode is stabilized, P would be stabilized in at most $\nu(P)$ steps. The final $K \in \mathcal{S} \cap \mathcal{T}^d$ that will stabilize P , is equal to the summation of controllers at each step, i.e.:

$$K(s) \triangleq \sum_k K_m^{(k)}(s).$$

We can easily show that stability of all the fixed modes of P , $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$, is also a necessary condition for the existence of stabilizing controller:

Theorem 21: A plant P is stabilizable by a controller $K \in \mathcal{S} \cap \mathcal{T}^d$, if and only if $\Lambda(P, \mathcal{S}, \mathcal{T}^s) \subset \mathbb{C}^-$.

Proof: The sufficiency part is done in Theorem 20. For the necessity part note that static fixed modes can not be moved by the dynamic controller either (Theorem 13), i.e.:

$$\begin{aligned} &\Lambda(P, \mathcal{S}, \mathcal{T}^s) \not\subset \mathbb{C}^- \\ &\stackrel{\text{Thm.13}}{\Rightarrow} \Lambda(P, \mathcal{S}, \mathcal{T}^d) \not\subset \mathbb{C}^- \\ &\stackrel{\text{by def}}{\Rightarrow} \nexists K \in \mathcal{T}^d \cap \mathcal{S} \text{ s.t. } A_{\text{CL}}(P, K) \subset \mathbb{C}^-. \end{aligned}$$

V. SYNTHESIS AND NUMERICAL EXAMPLE

In this section we provide an explicit algorithm to stabilize a plant which has no unstable fixed modes, and run it on one numerical example to illustrate its implementation. Algorithm 1 is distilled from the steps taken in the paper to prove the main theorem, and thus can almost certainly be improved upon in several respects.

In Algorithm 1, D is chosen randomly at each outer-step, and as stated in Remark 15, would be a valid choice with probability one. This D must be chosen small enough ($\|D\|_\infty < \gamma^{(k)}$) such that the total number of unstable modes would not increase when each element of the sequence $\{D|_{(m)}\}_{m=1}^a$ is closed around $G^{(k)}$. A prior knowledge of such an upper bound on D , denoted by $\gamma^{(k)}$, is not available and is hard to attain. This leads us to consider the alternative approach of repeatedly making D smaller in a loop until Theorem 16.2 holds. This iterative scaling repeats itself when (21) is not met. In this case, as proof suggests, we perturb $L^{(k)}$ by $\hat{L}^{(k)}$. This perturbation must be chosen small enough that it will not make any modes of $\tilde{A}_{11}^{(k)} - (L^{(k)} + \hat{L}^{(k)})\tilde{C}_1^{(k)}$ unstable. The upper bound on the perturbation $\hat{L}^{(k)}$ is unknown, and thus, similar to the case for D , we iterate to make it small enough to meet the constraints.

Remark 22: The intersection in the **if-then** section in Algorithm 1 would almost always result in a null set if interpreted with unlimited precision. However, choosing to replace the exact intersection with a proximity condition could possibly avoid very large feedback and observer gains.

Algorithm 1 Finding a controller $K \in \mathcal{T}^d \cap \mathcal{S}$ to stabilize P

Input: Plant P , information structure \mathcal{S}
Output: Controller $K \in \mathcal{T}^d \cap \mathcal{S}$ that will stabilize P
 $k \leftarrow 0, G^{(0)} \leftarrow P, K(\sigma) \leftarrow 0$

/* Repeat the outer loop until the plant is stabilized */

while $|\Lambda_+^\sim(G^{(k)})| \geq 1$ **do**

/* Select a static controller as in Rem. 15 */

 Choose a random $D \in \mathcal{T}^s \cap \mathcal{S}$
while $\nu(\Gamma(G^{(k)}, D)) > \nu(G^{(k)})$ **do**
 $D \leftarrow D/2$
end while

/* Find a controllable index as in Thm. 16 */

 $m^{(k)} \leftarrow a$
while $\Lambda_+^\sim(\Gamma(G^{(k)}, D|_{(m^{(k)}-1)})) \cap \Lambda_+^\sim(G^{(k)}) = \emptyset$ **do**
 $m^{(k)} \leftarrow m^{(k)} - 1$
end while

/* Form the SISO plant as in Fig. 4 */

 $G_{m^{(k)}}^{(k)} \leftarrow \mathbf{e}_{j_{m^{(k)}}}^T \Gamma(G^{(k)}, D|_{(m^{(k)}-1)}) \mathbf{e}_{i_{m^{(k)}}}$

/* Stabilize the SISO plant as in Prop. 17 */

 Find a Kalman similarity transformation $T^{(k)}$ for $G_{m^{(k)}}^{(k)}$

 Name all the corresponding partitions by $(\cdot)^{(k)}$

 Find a $F^{(k)}$ to stabilize $\tilde{A}_{11}^{(k)} - \tilde{B}_1^{(k)} F^{(k)}$

 Find a $L^{(k)}$ to stabilize $\tilde{A}_{11}^{(k)} - L^{(k)} \tilde{C}_1^{(k)}$

/* Ensuring that constraint (26) holds */

 $M^{(k)} \leftarrow (I - D|_{(m^{(k)}-1)} D_G^{(k)})^{-1}$
 $\check{D}^{(k)} \leftarrow \mathbf{e}_{j_{m^{(k)}}}^T D_G^{(k)} M^{(k)} \mathbf{e}_{i_{m^{(k)}}}$
if $\text{eig}_+(\tilde{A}_{11}^{(k)} - \tilde{B}_1^{(k)} F^{(k)} + L^{(k)} (\check{D}^{(k)} F^{(k)} - \tilde{C}_1^{(k)})) \cap$
 $\left(\bigcup_{i=2}^4 \text{eig}_+(\tilde{A}_{ii}^{(k)}) \right) \neq \emptyset$ **then**

/* Perturb the observer gain if (26) does not hold */

 Choose a random $L_\epsilon^{(k)}$

/* Make the perturbation sufficiently small not to have any new unstable mode */

while $|\text{eig}_+(\tilde{A}_{11}^{(k)} - (L^{(k)} + L_\epsilon^{(k)}) \tilde{C}_1^{(k)})| \geq 1$ **do**
 $L_\epsilon^{(k)} \leftarrow L_\epsilon^{(k)}/2$
end while
 $L^{(k)} \leftarrow L^{(k)} + L_\epsilon^{(k)}$
end if

/* Construct the MIMO controller as in Cor. 18 */

$$K^{(k)} \leftarrow \left[\begin{array}{c|c} \tilde{A}_{11}^{(k)} - \tilde{B}_1^{(k)} F^{(k)} + L^{(k)} (\check{D}^{(k)} F^{(k)} - \tilde{C}_1^{(k)}) & L^{(k)} \mathbf{e}_{j_{m^{(k)}}}^T \\ \hline -\mathbf{e}_{i_{m^{(k)}}} F^{(k)} & D|_{(m^{(k)}-1)} \end{array} \right]$$
 $K \leftarrow K + K^{(k)}$
 $G^{(k+1)} \leftarrow \Gamma(G^{(k)}, K^{(k)})$
 $k \leftarrow k + 1$
end while
return K

Remark 23: We can replace \mathbb{C}^- throughout the paper with another open set of acceptable closed-loop eigenvalues, letting its complement replace \mathbb{C}^+ as the closed set of unacceptable closed-loop eigenvalues. The results of Section IV-A hold up to show that the fixed modes must not be in the unacceptable region, the results of Section IV-B hold up to show that if they are not, then all of the modes can be moved to the acceptable region, and Algorithm 1 can be applied to find a controller which achieves that objective. One can further define a smaller open set of desirable closed-loop eigenvalues into which all of the non-fixed modes can be moved by Algorithm 1, taking note of the possibility of fixed and non-fixed modes overlapping in the acceptable-yet-undesirable region, as mentioned in Section II.

The following numerical example will use Algorithm 1 to stabilize the plant P .

Example 24: Consider the following plant:

$$A = \text{diag}(2, 3, 5, -1, -1)$$

$$B = \begin{bmatrix} 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 0 & 8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

and $D = 0$. Let the information constraint for the controller be given by the admissible-to-be-nonzero indices: $\text{Adm}(\mathcal{S}) = \{(1, 1), (3, 1), (4, 1), (5, 2), (1, 3), (3, 3), (4, 3), (5, 4), (5, 5)\}$. This plant has fixed mode $\Lambda(P, \mathcal{S}, \mathcal{T}^s) = \{-1\}$. If we follow Algorithm 1 to stabilize P , and choose our desired closed-loop modes of $\Gamma(P, K)$ to be $[-0.5 \ -1 \ -1 \ -1.5 \ -2 \ -2.5 \ -3 \ -3.5]^T$, this is achieved by the following resulting controller:

$$A_K = \begin{bmatrix} 14.9224 & -460.4053 & -4.6620 \\ 0.3742 & -24.4429 & 0.7485 \\ 22.9223 & -763.8424 & -25.4224 \end{bmatrix}$$

$$B_K = \begin{bmatrix} & 317.1161 & \\ 0_{3 \times 1} & 27.4429 & 0_{3 \times 3} \\ & 405.6193 & \end{bmatrix}$$

$$C_K = \begin{bmatrix} & 3.9020 & \\ 0_{3 \times 4} & -71.6446 & \\ & -7.4494 & \end{bmatrix}^T$$

$$D_K = \begin{bmatrix} 0.0854 & \\ 0 & \\ 0.4265 & 0_{5 \times 4} \\ 0.0936 & \\ 0 & \end{bmatrix}.$$

An alternative approach is taken in [12], in which, at each step, a (possibly dynamic) stabilizing controller is applied at the next diagonal element of the controller, and it is shown that by adding stabilizing controllers at each step, the set of (possibly unstable) fixed modes are reduced, until the last step where the remaining fixed modes must be necessarily stable. Applying the method of [12] on this plant would result in a stabilizing controller of order 7, as compared to 3 here. An explanation could be that in [12], a (possibly dynamic)

stabilizing controller is applied at each of the elements, resulting in abundant of controller states, whereas in here, only for each unstable mode, a stabilizing controller (not necessarily of order 1) is needed.

If we look at each of the nine SISO maps from u_{i_m} to y_{j_m} in P , then the union of controllable and observable modes of all these SISO maps are $\{2, 5\}$, which does not contain the unstable mode 3. This shows that a static gain (the D_{m-1} of Figure 4) might be necessary to assign some modes in decentralized settings, which is counter-intuitive compared to the centralized case where a stabilizing observer-based controller would have zero static gain.

VI. CONCLUSION

We revisited, verified, and generalized classic work in the stabilizability of decentralized systems. We generalized the notion of fixed modes to arbitrary information structure, and provided a rigorous inductive proof that plant modes which cannot be moved by static LTI controllers with the prescribed structure cannot be moved by dynamic LTI controllers either. We addressed the placement of the modes which are not fixed, and showed that they can be moved to within a chosen accuracy of any desired pole locations, thus solidifying and generalizing the main result of [1]. Combining these results, we have shown that having all fixed modes in the LHP with respect to static LTI controllers of a given information structure is necessary and sufficient for stabilizability by dynamic LTI controllers with the same structure. We lastly presented an explicit algorithm for finding such a stabilizing decentralized controller.

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